JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 31, No. 3, May–June 2008

Slewing of Flexible Spacecraft with Minimal Relative Flexible Acceleration

Alexandr E. Zakrzhevskii*
National Academy of Sciences of the Ukraine, 03057 Kiev, Ukraine

DOI: 10.2514/1.32215

An effective method for the optimal control for slewing maneuvers of a spacecraft with flexible appendages is presented. The dynamic equations of motion are formulated, allowing one to take into account the infinite number of natural vibration modes of the flexible appendages in a finite dimensional mathematical model. The problem of optimal reorientation for rest-to-rest maneuvers in the class of rotations with respect to Euler's axis is formulated, using an objective function that results in the minimal acceleration of the relative motion of the flexible elements during the maneuver. The new features and advantages of the proposed approach are the use of new dynamic equations of motion of elastic systems and an infrequently used objective function for the optimal control problem, which has a clear physical interpretation. A general formulation of the problem is given and some special cases are considered. Solutions are obtained for the special cases. For the first case, the numerical solution based on the developed solution algorithm for the nonlinear boundary-value problem is presented. For the second case, a simple analytical solution is obtained.

I. Introduction

IN RECENT years, there has been an increased interest in the attitude control of elastic spacecraft (in particular, in their optimal reorientation). In practice, control of such systems uses a combination of feedforward and feedback control. Measurement of the deformation of flexible appendages is possible only when distributed sensors are used. The performance of feedback is limited if a spacecraft does not have such sensors. Thus, feedforward control, which takes proper account of the object's dynamic properties, is of great importance for this problem.

Optimization of a rest-to-rest maneuver of a flexible spacecraft can entail various objectives. For spacecraft on long missions with a weak power-to-mass ratio, an important objective is to minimize the energy cost [1–4]. For spacecraft dealing with astronomical observations and performing a great number of reorientation maneuvers, it may be important to minimize time. Time-optimal rest-to-rest slewing of flexible systems has been investigated by several authors [5–14]. Typically, the solution of the problem results in bang–bang control, which can be very sensitive to system modeling errors. Often, these objectives are combined into a single cost function with weighted items.

For some cases, minimization of the energy cost or the transfer time is not as important as the minimization of the perturbation of the main body's attitude motion by flexible vibrations. The slewing rest-to-rest maneuver of such spacecraft needs to be controlled so that practically no flexible vibrations of their appendages are excited during the maneuver. This will be the problem considered in this paper.

The first attempts to formulate an objective function that takes into account design elastic properties were made in the seventies. In 1972, Zakrzhevskii [15,16] used a quadratic objective function, the physical meaning of which was to minimize the relative acceleration of the flexible elements during the controlled motion. In 1979, Farrenkopf [17] used a quadratic criterion that included the terminal value and the mean value of the generalized elastic coordinates and their first and second derivatives in the cost function. Other investigations in this area include a number of approaches developed for linear flexible systems, which shape the feedforward input such that it does not contain spectral components at system eigenfrequencies [18]. Modifications of such methods have been applied to nonlinear flexible systems [19], but they may yield a significant level of residual vibrations [20].

Gorinevsky and Vukovich [20] used the weighted sum of squares of the norm of the terminal error and the feedforward shape vector as an objective function. The minimization of the second term closely represents the fuel optimality condition. The approach to solve such problems, especially the minimal-time problem, is based on the use of the maximum principle. All these papers treat only elastic systems discretized by one mode. In this paper, the generalization to an arbitrary number of elastic modes is given. The paper presents a widely unknown approach for formulating the equations of motion of a spacecraft with attached flexible appendages and for posing the optimal control problem for its slewing maneuver. The basic idea behind this approach [21] is to divide the infinite system of Lagrange's equations for flexible appendages into two groups. The equations of the first group are ordinary differential equations of the second order. They conform to the usual approach to investigate such systems. The equations of the second group degenerate into algebraic equations. They relate to quasi-static behavior of the appropriate generalized coordinates. The border between these groups is determined by the condition of quasi-static deformation, which is introduced here. This approach permits one to build a finite dimensional system of equations of motion for a continuous system, which takes into account the interaction of the spacecraft's main body with any number of modes of the flexible appendages. Naturally, taking into account an infinite number of modes has limited usefulness in practical applications, but it permits generalizing known models and obtaining an estimate of the truncation errors.

In the formulation of the optimal control problem, the objective function serves to minimize the relative acceleration of the flexible appendages during the maneuver. This allows controlling the elastic deformations as close as possible to the quasi-static relative motion, performing the spacecraft's slewing maneuver with respect to the Euler axis.

Three types of models are considered: 1) a general model with a finite number of elastic modes, with the remaining modes being quasi-static; 2) a model with a finite number of elastic modes, with the remaining modes neglected; and 3) a model in which all elastic modes are quasi-static. For the first (general) case, the full

Received 17 May 2007; accepted for publication 6 November 2007.

*Leading Scientist, Polyaggregate Systems' Dynamics Department, S. P. Timoshenko Institute of Mechanics, 3 Petra Nesterova Street.

mathematical model is formulated. For the second (special) case, an efficient iterative algorithm of numerically solving the appropriate nonlinear boundary-value problem is developed. A code in Fortran has been written and tested for the example given in [20]. The third (special) case results in a linear model and the optimal control problem can be solved analytically.

II. Problem Formulation

A. Mechanical Model and Equations of Motion

The model describing the dynamics of the flexible appendages is based on the following assumptions:

1) The appendages are identical and symmetrically attached with respect to the mass center of the rigid main body. They can be treated as uniform beams, which can perform oscillations in the plane x_1Ox_3 (Fig. 1).

2) The bending stiffness of the appendages in plane x_2Ox_3 is very high, and their deformation in this plane can be neglected.

3) Torsional deformations are not taken into account (we include this assumption for brevity; all results for the bending vibrations can also be easily extended to torsional vibrations).

4) All external moments are negligible compared with the control moments.

5) The vector of the flexible relative displacements of the appendages can be written as a series [22]:

$$\boldsymbol{u}_j = \sum_{\alpha=1}^{\infty} q_{\alpha,j}(t)\boldsymbol{U}_{\alpha,j}(x_3) \qquad (1)$$

where $q_{\alpha,j}(t)$ are the unknown generalized coordinates, and $\boldsymbol{U}_{\alpha,j}(x_3)$ are natural modes for a cantilevered beam. In Eq. (1), the index $j = 1$, 2 corresponds to the number of flexible appendages specified in Fig. 1.

6) The mass and stiffness characteristics are symmetric with respect to the main planes of inertia of the undisturbed system.

Under these assumptions, the equations of motion for the new generalized coordinates $\zeta_\alpha = -q_{\alpha 1} + q_{\alpha 2}$ and $\xi_\alpha = q_{\alpha 1} + q_{\alpha 2}$, in accordance with [22] (Sec. 9.10), may be written as

$$\begin{gathered}\ddot{\zeta}_\alpha + (\Omega_\alpha)^2\zeta_\alpha = (\omega_1\omega_3 + \dot{\omega}_2)t_{\alpha 2}(t_{\alpha 4})^{-1} + \left(\omega_2^2 + \omega_3^2\right)\zeta_\alpha \\ \ddot{\xi}_\alpha + (\Omega_\alpha)^2\xi_\alpha = 0 \qquad (\alpha = 1, \ldots, \infty)\end{gathered} \qquad (2)$$

where Ω_k are the natural frequencies of flexural vibrations of the cantilevered beam, ω_i $(i = 1, 2, 3)$ are the main-body angular-velocity components

$$t_{\alpha 2} \triangleq m_p L \int_0^1 |\boldsymbol{U}_\alpha(s)|(s + RL^{-1})\,\mathrm{d}s$$

and

$$t_{\alpha 4} \triangleq m_p \int_0^1 (\boldsymbol{U}_\alpha(s))^2\,\mathrm{d}s$$

m_p is the mass of each appendage, L is its length, and s is the dimensionless longitudinal coordinate of the beam. As a result of the action of the control moments on the main body, linear combinations of the generalized coordinates $\zeta_\alpha = -q_{\alpha 1} + q_{\alpha 2}$ are only excited.

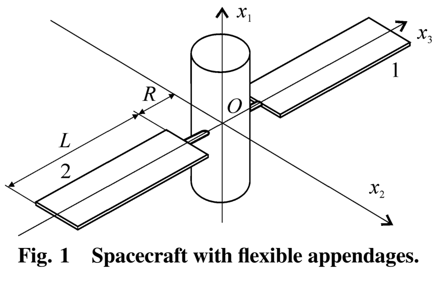

Fig. 1 Spacecraft with flexible appendages.

Now we consider the kinematics of the Eulerian slew for the description of the programmed motion of the main body. The following frames of reference are used in the formulation of the equations of motion. $Ox_1^0x_2^0x_3^0$ and $Ox_1^fx_2^fx_3^f$ are attached to the inertial reference frames coincident with $Ox_1x_2x_3$ at the beginning and at the end of the slew, respectively. $O\xi_1\xi_2\xi_3$ is fixed in the main module reference frame and is coincident at the beginning of the slew with $O\xi_1^0\xi_2^0\xi_3^0$, so that the axes $O\xi_1$ and $O\xi_1^0$ coincide with the Eulerian axis. Now we relate the system of unit vectors $\boldsymbol{i}_k$, $\boldsymbol{i}_k^0$, $\boldsymbol{e}_k$, $\boldsymbol{e}_k^0$, and $\boldsymbol{i}_k^f$ to the frames of reference Ox_k, Ox_k^0, $O\xi_k$, $O\xi_k^0$, and Ox_k^f $(k = 1, 2, 3)$, respectively. The preceding coordinate frames are related to each other through the transformation matrices:

$$\begin{gathered}\boldsymbol{i}_k = \sum_{j=1}^{3} c_{kj}^E \boldsymbol{i}_j^0; \qquad \boldsymbol{e}_k^0 = \sum_{m=1}^{3} \gamma_{km}\boldsymbol{i}_m^0 \\ \boldsymbol{e}_k = \sum_{m=1}^{3} \gamma_{km}\boldsymbol{i}_m^0 = \sum_{m=1}^{3} b_{km}^E \boldsymbol{e}_m^0; \qquad \boldsymbol{i}_k^T = \sum_{m=1}^{3} \alpha_{km}^E \boldsymbol{i}_m\end{gathered} \qquad (3)$$

The matrix $\boldsymbol{b}^E$, for example, is given by

$$\boldsymbol{b}^E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix}$$

where $\psi(t)$ is the slewing angle of the basis $O\xi_k$ with respect to the axis $O\xi_1$. The matrix α^E is determined by the given angles of the slew with respect to axes, which are connected with the main module. They allow the determination of the value of the angle of the finite slew Φ with respect to the axis $O\xi_1$ and the direction cosines γ_{1i} of the axis $O\xi_1$ in the frame of reference Ox_i^0 $(i = 1, 2, 3)$ as

$$\gamma_{11} = \frac{a_{32}^E - a_{23}^E}{2\sin\Phi}; \qquad \gamma_{12} = \frac{a_{13}^E - a_{31}^E}{2\sin\Phi}; \qquad \gamma_{13} = \frac{a_{21}^E - a_{12}^E}{2\sin\Phi} \qquad (4)$$

These direction cosines are identical in the frames of reference Ox_k^0, Ox_k, and Ox_k^f $(k = 1, 2, 3)$. The angle of a finite rotation Φ is determined in accordance with the formula $\Phi = \arccos\frac{1}{2}(Sp\alpha^E - 1)$, where

$$Sp\alpha^E = \sum_{i=1}^{3} a_{ii}^E$$

is the trace of the matrix α^E.

The direction cosines of the vectors $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ can be chosen such that the vectors $\boldsymbol{e}_i$ form a right-hand coordinate basis. Choose an axis $O\xi_2$ in the intersection of a plane having the vector $\boldsymbol{e}_1$ as normal and a plane having the vector $\boldsymbol{i}_1$ as normal. Having chosen a plus sign before the normalizing square root, one can write

$$\gamma_{21} = 0; \qquad \gamma_{22} = \gamma_{13}\sqrt{\gamma_{12}^2 + \gamma_{13}^2}; \qquad \gamma_{23} = -\gamma_{12}\sqrt{\gamma_{12}^2 + \gamma_{13}^2} \qquad (5)$$

From the condition $\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$, it follows that

$$\begin{gathered}\gamma_{31} = \gamma_{12}\gamma_{23} - \gamma_{22}\gamma_{13}; \qquad \gamma_{32} = \gamma_{13}\gamma_{21} - \gamma_{11}\gamma_{23} \\ \gamma_{33} = \gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21}\end{gathered} \qquad (6)$$

There are various possibilities to choose the components of the matrix γ. From the condition that the $\boldsymbol{e}_i$ $(i = 1, 2, 3)$ form a right-handed set of three mutually orthogonal vectors, we have

$$c_{ki} = \sum_{p=1}^{3}\sum_{q=1}^{3} \gamma_{pk} b_{pq} \gamma_{ql} \qquad (k, l = 1, 2, 3) \qquad (7)$$

Hence, the direction cosines of the system of axes fixed in the main body can be defined as a function of the angle $\psi(t)$. The angular-velocity vector of the main body coincides with the axis $O\xi_1$ during the Eulerian slew [i.e., $\boldsymbol{\omega} = \boldsymbol{e}_1\dot{\psi}(t)$] or

$$\omega_i = \dot{\psi}(t)\,\gamma_{1i} \qquad (i = 1, 2, 3) \tag{8}$$

Hence, the rotational motion of the main body is described by one kinematic parameter ψ.

Formally taking into account structural damping by introducing terms that are proportional to the appropriate generalized velocity, one can rewrite the first of Eq. (2) as follows:

$$\ddot{\zeta}_\alpha + n_\alpha \dot{\zeta}_\alpha + (\Omega_\alpha)^2 \zeta_\alpha = \dot{\psi}^2 \gamma_{11}\gamma_{13} t_{\alpha 2}/t_{\alpha 4} + \dot{\psi}^2 \zeta_\alpha (\gamma_{12}^2 + \gamma_{13}^2) + \ddot{\psi}\gamma_{12} t_{\alpha 2}/t_{\alpha 4} \qquad (\alpha = 1, 2, \ldots, \infty) \tag{9}$$

Let $u \triangleq \ddot{\psi}$ be the control variable and $z_1 \triangleq \psi$ and $z_2 \triangleq \dot{\psi}$ be additional variables. Replace the variables ζ_α by variables $z_{\alpha+2}$. It is now possible to write the main-body equations of motion with respect to the Eulerian axis as

$$\dot{z}_1 = z_2; \qquad \dot{z}_2 = u \tag{10}$$

$$\dot{z}_{2k+1} = z_{2k+2}$$
$$\dot{z}_{2k+2} = -n_k z_{2k+2} - \Omega_k^2 z_{2k+1} + b_k z_2^2 + d_2 \cdot z_2^2 z_{2k+1} + a_k u \qquad (k = 1, 2, \ldots, \infty) \tag{11}$$

where $b_k = -\gamma_{11}\gamma_{12} t_{k2}/t_{k4}$, $a_k = \gamma_{12} t_{k2}/t_{k4}$, and $d_2 = \gamma_{12}^2 + \gamma_{13}^2$.

B. Model Reduction

In mathematical models of flexible spacecraft, researchers usually consider only a small number of modes of flexible vibrations (frequently only one), without proper justification and without an error estimate of such a truncation. We will show, however, how a finite dimensional truncation of the continuous system based on an error estimation can be performed.

We introduce into the analysis three groups of characteristic time scales: the characteristic periods $T_{1\alpha}$ of the natural flexible vibrations; the characteristic times $T_{2\alpha}$, which relate to the periods after these oscillations are extinguished by damping; and the characteristic time T_3 of the spacecraft attitude motion. In particular, T_3 may be taken to be $|\boldsymbol{\omega}|^{-1}$ for stationary motions. For terminal motions such as a programmed slewing maneuver, it is possible to consider T_3 to be the slew duration. Because possible values of T_3 are finite and because the values of $T_{1\alpha}$ and $T_{2\alpha}$ decrease with increasing values of α, it is possible to assume that the conditions

$$T_{1\alpha} \ll T_{2\alpha} \ll T_3 \tag{12}$$

are satisfied for the introduced time scales starting with some number $\alpha = N + 1$ and that the sets $T_{1\alpha}$ and $T_{2\alpha}$ are monotonously decreasing sequences.

Owing to conditions (12), natural vibrations of flexible elements for the mode with the number $\alpha = N + 1$ damp out during $T_{2\alpha} \ll T_3$. The same is true for all modes with $\alpha > N + 1$. Therefore, in the study of the motion of the system on intervals of time of the order $T_{2\alpha}$ and greater, natural vibrations for all modes for $\alpha > N$ can be neglected.

Let us assume in Eqs. (11) that

$$\Omega_\alpha^2 = \varepsilon_\alpha^{-2}\left(\Omega_\alpha^0\right)^2; \qquad n_\alpha = \delta_\alpha \varepsilon_\alpha^{-1} n_\alpha^0 \qquad (\alpha = N + 1, \ldots, \infty) \tag{13}$$

where Ω_α^0 and n_α^0 are limited modified values of Ω_α and n_α, respectively, and ε_α and δ_α are dimensionless small parameters determined by the conditions

$$0 < \varepsilon_\alpha \ll \delta_\alpha \ll 1; \qquad \varepsilon_\alpha > \varepsilon_{\alpha+1}$$
$$\delta_\alpha > \delta_{\alpha+1} \qquad (\alpha = N + 1, \ldots, \infty) \tag{14}$$

In the case when $\varepsilon_\alpha \to 0$, which corresponds to increasing the stiffness to infinity, it follows that $z_{2k+1} \equiv 0$ $(k = N + 1, \ldots, \infty)$ for all higher flexible modes. This corresponds to the conventional technique of considering only a finite number of flexible modes in the mathematical model.

In the case of small positive values ε_α and δ_α, Eq. (11) can be written in the following form:

$$\varepsilon_k^2 \ddot{z}_{2k+1} + \varepsilon_k \delta_k n_k^0 \dot{z}_{2k+1} + \left(\Omega_k^0\right)^2 z_{2k+1} = \varepsilon_k^2 \Big(b_k z_2^2 + d_2 \cdot z_2^2 z_{2k+1} + a_k u\Big) \qquad (k = N + 1, \ldots, \infty) \tag{15}$$

The approximate solution of Eq. (11) with small parameters at the terms of first and second derivatives can be found by asymptotic methods as the sum of a regular part and the boundary-layer solution, which is rapidly damped from the initial moment of time (i.e., at values of time of the order $T_{2\alpha}$ and greater). We find the regular part of the solution of Eq. (12) with respect to ε_α and δ_α as a series expansion of powers of the values ε_α^2 and $\varepsilon_\alpha \delta_\alpha$. Taking into account the inequalities (14), one obtains

$$z_{2k+1} = \varepsilon_k^2 z_{2k+1}^{(0)} + \varepsilon_k^3 \delta_k z_{2k+1}^{(1)} + O\left(\varepsilon_k^4\right) \qquad (k = N + 1, \ldots, \infty) \tag{16}$$

In particular, we can choose such values ε_α as the following small parameters:

$$\varepsilon_k = T_{1k} T_3^{-1} = 2\pi\, \Omega_k^{-1} T_3^{-1} \tag{17}$$

After substituting the expansion (16) in Eq. (15) and equating coefficients of identical degrees of the parameters ε_α and δ_α, one obtains

$$z_{2k+1} = (\Omega_k)^{-2}[Q_{*k} - n_k(\Omega_k)^{-2}\dot{Q}_{*k}] + O(\varepsilon_k^4) \tag{18}$$

where $Q_{*k} = b_k z_2^2 + a_k u$, $(k = N + 1, \ldots, \infty)$.

Equation (18) describe a slow forced motion, because natural vibrations for these modes are described by the boundary-layer solution, which is rapidly damped beginning from the initial moment of time. Taking into account Eq. (17), one may rewrite Eq. (18) as follows:

$$z_{2k+1} = (\Omega_k)^{-2} Q_{*k} + O\left(\varepsilon_k^4\right) \tag{19}$$

C. Formulation of the Optimal Control Problem

We want to transfer the spacecraft from the initial conditions

$$z_1(0) = -\Phi; \qquad z_j(0) = 0 \qquad (j = 2, \ldots, \infty) \tag{20}$$

to the terminal conditions

$$z_i(T) = 0 \qquad (i = 1, 2, \ldots, \infty) \tag{21}$$

with the minimal effect of the relative motion of flexible appendages on the main body during time T.

The most crucial element of the statement of the optimal control problem is the choice of the objective function. In the case under consideration, the objective function needs to be chosen so as to satisfy the requirement of minimizing the effect of the relative motion of the flexible elements on the attitude of the main body. Basically, it is necessary to minimize all terms in the right-hand side of the equations of the attitude motion of the main body, which depend on coordinates introduced from flexibility. The solution of this problem is nontrivial, although the objective functions have a precise physical interpretation in some cases.

According to Balchen [23], who considered that a system must be optimized by a vectorial criterion instead of a scalar criterion, it is possible to minimize vectorial terms in the equations of the attitude motion of the system, which determine the effect of the flexible elements on the main body (in notations of [22]) as

$$\boldsymbol{K}_r^{*C} + \boldsymbol{\omega} \times \boldsymbol{K}_r^C \tag{22}$$

where $\boldsymbol{K}_r^C$ is the main relative moment of momentum of the flexible elements with respect to the mass center C.

In the equations of the attitude motion of the main body, the term (22) in notations of [22] (Sec. 9.9) looks like

$$\sum_{\alpha=1}^{\infty} q_\alpha[2(\boldsymbol{\Lambda}^\alpha \cdot \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{\Lambda}^\alpha \cdot \boldsymbol{\omega})] + \sum_{\alpha=1}^{\infty} \ddot{q}_\alpha(2\boldsymbol{\omega} \cdot \boldsymbol{\Lambda}^\alpha + \boldsymbol{\omega} \times \boldsymbol{G}^\alpha) + \sum_{\alpha=1}^{\infty} \ddot{q}_\alpha \boldsymbol{G}^\alpha \tag{23}$$

The choice of the norm of the vectorial value (23) as a subintegral in the objective function allows satisfying the requirements without problems. However, the optimal control problem reduces to a nonlinear boundary-value problem, which is difficult to solve. Because the control objective consists of decreasing the attitude perturbation of a main body within given limits, it is possible to try to solve this problem with a simpler objective function, which nevertheless reflects the main requirements on the motion of the system. Studying the behavior of the real system at the obtained feedforward control will show whether it is necessary to use a more complicated objective function.

In practice, the case when $|\boldsymbol{\omega}| \ll 1\ \mathrm{s}^{-1}$ is of prime interest. In this case, the dominant effect of a flexible element on the main body in expression (23) is determined by the linear term, which is proportional to $\ddot{q}_\alpha$.

We choose an objective function close to the aforementioned requirement and, at the same time, simple enough for the solution of the boundary-value problem. Such an objective function may be represented by the following functional:

$$I = \frac{1}{2}\int_0^T \sum_{k=1}^{\infty} \dot{z}_{2k+2}^2 w_k \,\mathrm{d}t \tag{24}$$

This functional can be interpreted as an objective function that minimizes dynamic acceleration loads on flexible elements in relative motion, where w_k are some weight coefficients. They can have various physical meanings. In case it is necessary to isolate only the main body from low-frequency perturbations, the corresponding w_k can be set equal to unity, and the remaining w_k can be set to zero. Such an approach corresponds to the conventional truncation of the mathematical model to a finite number of flexible modes.

The equations of motion (10) and (11) act as constraints on the state variables z_k. Equation (10) corresponds to the integral constraint

$$\int_0^T \int_0^t u(\tau)\,\mathrm{d}\tau\,\mathrm{d}t = \Phi \tag{25}$$

One can write now the optimal control problem as follows:

$$\begin{gathered}\dot{z}_1 = z_2; \qquad \dot{z}_2 = u; \qquad \dot{z}_{2k+1} = z_{2k+2} \\ \dot{z}_{2k+2} = -n_k z_{2k+2} - \Omega_k^2 z_{2k+1} + b_k z_2^2 + d z_2^2 z_{2k+1} + a_k u \\ (k = 1, 2, \ldots, \infty)\end{gathered} \tag{26}$$

$$I = \frac{1}{2}\int_0^T \left[\sum_{k=1}^{N} \dot{z}_{2k+2}^2 w_k + \sum_{k=N+1}^{\infty} \Omega_k^{-4}\Big[2b_k(u^2 + z_2\dot{u}) + a_k^2\ddot{u}\Big]^2 w_k\right]\mathrm{d}t \tag{27}$$

The boundary conditions of the problem look like Eqs. (20) and (21).

Let us introduce additional state variables $x_3 \triangleq u$ and $x_4 \triangleq \dot{u}$, and a new control variable $v \triangleq \ddot{u}$. We change variables z_{2k+1} and z_{2k+2} to x_{2k+3} and x_{2k+4}. If condition (12) is satisfied starting with some $k = N + 1$, it is enough to keep $2N$ equations in system (11) and to use solutions (19) for the remaining equations. In the new variables, the optimal control problem is formulated as follows:

$$\begin{gathered}\dot{x}_1 = x_2; \qquad \dot{x}_2 = x_3; \qquad \dot{x}_3 = x_4; \qquad \dot{x}_4 = v \\ \dot{x}_{2k+3} = x_{2k+4} \\ \dot{x}_{2k+4} = -n_k x_{2k+4} - \Omega_k^2 x_{2k+3} + b_k x_2^2 + d_2 x_2^2 x_{2k+3} \\ + a_k x_3 \qquad (k = 1, 2, \ldots, N)\end{gathered} \tag{28}$$

$$I^* = \frac{1}{2}\int_0^T \left[\sum_{k=1}^{N} \dot{x}_{2k+4}^2 w_k + \sum_{k=N+1}^{\infty} \Omega_k^{-4}\Big[2b_k\Big(x_3^2 + x_2 x_4\Big) + a_k^2 v\Big]^2 w_k\right]\mathrm{d}t \tag{29}$$

The system of Eq. (28) differs from Eq. (26) not only in the sense that it contains a finite number of state variables, but it also contains the additional state variables x_3, x_4. Formulation of the boundary conditions for these variables is caused by relation (19). When conditions (20) and (21) are satisfied, the boundary conditions for x_3 and x_4 are homogeneous. In other cases, the boundary conditions for x_3 and x_4 are determined by $x_j(0)$ and $x_j(T)$ for $j > N$. Thus,

$$\begin{gathered}x_1(0) = -\Phi; \qquad x_j(0) = 0 \qquad (j = 2, \ldots, 2N + 2) \\ x_i(T) = 0 \qquad (i = 1, \ldots, 2N + 4)\end{gathered} \tag{30}$$

The infinite series, which are part of the expression (29), converge rapidly because $a_k \geq a_{k+1}$and $w_k = \mathrm{const}$ for the normalized modes of flexible vibrations and because the series

$$\sum_{k=N+1}^{\infty} \Omega_k^{-4}$$

which is reduced to the series

$$\sum_{k=N+1}^{\infty} (k - 0.5)^{-8}$$

up to a constant multiplier, also converges rapidly.

Note that the problem under consideration includes the special case when $\gamma_{12} = \gamma_{13} = 0$. In this case, the Eulerian axis coincides with Ox_1. The control variable in Eq. (26) disappears and this problem becomes meaningless. Assume further that γ_{12} and γ_{13} are not equal to zero simultaneously (i.e., $\gamma_{11} \neq 1$). Concerning the controllability of the system, we note that the linearized system is controllable if 1) $\gamma_{11} \neq 1$, 2) $\Omega_i \neq 0$, $(i = 1, \ldots, \infty)$, and 3) $\Omega_i \neq \Omega_k$ $(i \neq k)$.

III. Optimization of Slewing Maneuvers of a Flexible System

A. Special Case I

As a first special case of the problem, we consider the case when we can neglect the terms determined by quasi-static solutions in the objective function (29). This problem corresponds to the minimization of dynamic acceleration loads only for the first N flexible modes, which make the most essential contribution to force effects of the flexible elements on the main body. The optimal control problems (28–30) can now be written as

$$\begin{gathered}\dot{x}_1 = x_2; \qquad \dot{x}_2 = u; \qquad \dot{x}_{2k+1} = x_{2k+2} \\ \dot{x}_{2k+2} = -n_k x_{2k+2} - \Omega_k^2 x_{2k+1} + d_2 x_2^2 x_{2k+1} + b_k x_2^2 + a_k u \\ (k = 1, \ldots, N)\end{gathered} \tag{31}$$

$$I = \frac{1}{2}\int_0^T \sum_{k=1}^{N}\Big(-n_k x_{2k+2} - \Omega_k^2 x_{2k+1} + d x_2^2 x_{2k+1} + b_k x_2^2 + a_k u\Big)^2 w_k\,\mathrm{d}t \tag{32}$$

$$x_1(0) = -\Phi; \qquad x_j(0) = 0 \qquad (j = 2, \dots, 2N+2)$$
$$x_k(T) = 0 \qquad (k = 1, \dots, 2N+2) \tag{33}$$

This is the Lagrange problem of the conditional extremum of the functional (32) with the constraints (31). A solution of the problem can be only obtained using numerical methods. For problems similar to this, one of the quasi-linearization methods is most effective. However, to start the iterative process, we need an approximation of the solution. For example, we may take the solution of the linearized problem.

Now we reduce the optimal control problem to a two-point nonlinear boundary-value problem. The Hamiltonian takes the form

$$H = L + \sum_{k=1}^{2N+2} \lambda_k f_k = \lambda_1 x_2 + \lambda_2 u + \sum_{k=1}^{N} \Big[\lambda_{2k+1} x_{2k+2} + \lambda_{2k+2} \Big(-n_k x_{2k+2} - \Omega_k^2 x_{2k+1} + b_k x_2^2 + x_2^2 x_{2k+1} d_2 + a_k u \Big) + \frac{1}{2} \Big(-n_k x_{2k+2} - \Omega_k^2 x_{2k+1} + b_k x_2^2 + x_2^2 x_{2k+1} d_2 + a_k u \Big)^2 w_k \Big] \tag{34}$$

where L is the Lagrangian function, λ_k are costate variables, and f_k are the right-hand members of Eq. (31) with the number k. The costate equations can be written as follows:

$$\dot{\lambda}_1 = 0;$$
$$\dot{\lambda}_2 = -\lambda_1 - 2 \sum_{i=1}^{N} \lambda_{2i+1} x_2 (b_1 + x_{2i+1} d_2) - \sum_{i=1}^{N} (-n_i x_{2i+2} - \Omega_i^2 x_{2i+1} + b_i x_2^2 + x_2^2 x_{2i+1} d_2 + a_i u) 2 x_2 (b_i + x_{2i+1} d_2) w_i$$
$$\dot{\lambda}_{2k+1} = \lambda_{2k+2} (\Omega_k^2 - x_2^2 d_2) + (\Omega_k^2 - x_2^2 d_2)(-n_i x_{2i+2} - \Omega_k^2 x_{2k+1} + b_k x_2^2 + x_2^2 x_{2k+1} d_2 + a_k u) w_k;$$
$$\dot{\lambda}_{2k+2} = -\lambda_{2k+1} + n_k \lambda_{2k+2} + n_k (-n_k x_{2k+2} - \Omega_k^2 x_{2k+1} + b_k x_2^2 + x_2^2 x_{2k+1} d_2 + a k_i u) w_k \tag{35}$$

From the necessary condition of optimality $\partial H / \partial u = 0$, one obtains the expression for the control variable as a function of the state and costate variables:

$$u = -\left(\sum_{i=1}^{N} a_i^2 w_i \right)^{-1} \left[\lambda_2 + \sum_{k=1}^{N} \lambda_{2k+2} a_k + \sum_{k=1}^{N} \Big(-n_k x_{2k+2} - \Omega_k^2 x_{2k+1} + b_k x_2^2 + x_2^2 x_{2k+1} d_2 \Big) w_k a_k \right] \tag{36}$$

Now the optimal control problem can be reduced to a two-point boundary-value problem for the nonlinear system of $4(N+1)$ ordinary differential equations of the first order:

$$\dot{x}_1 = x_2; \qquad \dot{x}_2 = u(x); \qquad \dot{x}_{2k+1} = x_{2k+2}$$
$$\dot{x}_{2k+2} = G_{1k}(x) + a_k u(x); \qquad \dot{x}_{N+1} = 0$$
$$\dot{x}_{N+2} = -x_{N+1} - \sum_{i=1}^{N} G_{2i}(x) [x_{M+2i+2} + G_{1i}(x) + a_i u(x)] w_i$$
$$\dot{x}_{M+3+2k} = -G_{3k}(x) [x_{M+3+2k} + G_{1k}(x) + a_k u(x)] w_k$$
$$\dot{x}_{M+4+2k} = -x_{M+3+2k} \qquad (k = 1, \dots, N) \tag{37}$$

where

$$M = 2N + 2$$
$$G_{1k}(x) = -n_k x_{2k+2} - \Omega_k^2 x_{2k+1} + b_k x_2^2 + x_2^2 x_{2k+1} d_2$$
$$G_{2k}(x) = 2(b_k x_2 + x_2 x_{2k+1} d_2); \qquad G_3(x) = -\Omega_k^2 + x^2 d \tag{38}$$
$$u(x) = \left(\sum_{i=1}^{N} a_i^2 w_i \right)^{-1} \left\{ x_{M+2} + \sum_{i=1}^{N} [x_{M+2+2i} + G_{1i}(x) w_i] \right\}$$

with the boundary conditions (33).

Equation (37) can be rewritten as

$$\dot{x}_s = \varphi_s(x) \qquad (s = 1, \dots, 2M) \tag{39}$$

The boundary conditions of the problem are

$$x_i(0) = 0 \qquad (i = 1, \dots, M); \qquad x_1(T) = \Psi$$
$$x_j(T) = 0 \qquad (j = 2, \dots, M) \tag{40}$$

One of the most relevant methods for solving such a problem is the successive linearization method [24]. The transitional matrix method [25] can be used here too.

The solution can be represented as

$$x^{(i+1)}(t) = x^{(i)}(t) + y^{(i)}(t) \qquad (i = 0, 1, \dots) \tag{41}$$

We can use the solution of the boundary-value problem obtained from Eqs. (39) and (40) by neglecting the nonlinear terms in the starting approximation $x^{(0)}(t)$. The linearized equations for all subsequent approximations can be obtained by the standard Taylor series expansion of the right-hand sides of Eq. (39) in the neighborhood of the solution obtained from the previous approximation, retaining all linear terms.

Recurrent relations for the appropriate equations become

$$\dot{y}_s^{(n)} - \sum_{p=1}^{2M} g_{sp}^{(n)} y_p^{(n-1)} = -\varepsilon \Big[\dot{x}_s^{(n)} - \varphi_s(x^{(n)}) \Big] \qquad (s = 1, \dots, 2M)$$
$$g_{sp}^{(n)} = \partial \varphi_s / \partial x_p |_{x = x^{(n)}} \tag{42}$$

The parameter ε is introduced here to speed up the convergence of the computation process. Its value should be gradually increased, finally approaching unity in the course of the successive approximations [24].

The resulting set of the equations for all approximations, except for the initial one, becomes a set of inhomogeneous equations with variable coefficients. The boundary conditions (40) remain the same in all iterations by virtue of their linearity. They are

$$y^{(n)}(t_0) = 0; \qquad y^{(n)}(T) = -\varepsilon [x^{(n)}(T) - x(T)] \tag{43}$$

Solving of the optimal control problem determines the motion of the main body that consists of its rotation with respect to the Eulerian axis under the law determined by $\psi(t)$.

As an example to illustrate the numerical procedure and to show the effectiveness of the proposed algorithm, we consider the optimal Euler-axis maneuver problem for a spacecraft with values of parameters given in [20]: length of flexible panels $L = 30$ m, linear density of appendages $\mu = 0,2$ kg/m, radius of the point of attaching the appendage $r = 1$ m, bending stiffness $EJ = 1500\ \mathrm{N/m^2}$, and nominal duration of the slewing maneuver $T = 25$ s. The decrement of oscillation $\vartheta_k = \pi n_k / \Omega_k$ was within the range of 0.02–0.05. This is the case when the duration of the maneuver is close to the vibration period of the system in its lowest mode. The number of flexible modes taken into consideration is two. The solution of the nonlinear boundary-value problem in this case needed a large number of iterations to provide convergence of the repetitive process. For computation of the optimal spacecraft slewing program appropriate to three serial rotations of the angles $\varphi_i =$

1.2 rad ($i = 1, 2, 3$) at $T = 25$ s, 20 iterations were required. After $T = 22$ s, the high accuracy of satisfying the terminal boundary conditions (errors were about 1.E-10 for all variables) and the stationarity of the functional (difference of its values on the successive iterations of the same order) were achieved only after 30 iterations. The computing process was diverging for the smaller values of T. For $T = 21$ s, it was impossible to have a converging computing process at all.

The results of computing the programmed slewing for its various duration ($T \in [22, 34]$ s) are shown in Figs. 2–5. As one can see from Fig. 5, the complexity of the feedforward control u increases as the duration of the slewing maneuver approaches the period of natural vibrations for the first mode. Also, the amplitudes of the perturbation of the second mode grow. Accordingly, the behavior of the angular velocity of the main body in Fig. 3 becomes more complicated because the perturbations from the second mode become more significant. The number of flexible modes taken into consideration in the model of the spacecraft under action of the found feedforward control is taken to be three. The first mode gives the greatest contribution in the maximum tip deflection (1.7 m at $T = 22$ s and 0.75 m at $T = 34$ s). The second mode yields 0.015 and 0.003 m, respectively. The contribution of the third mode is negligible (Fig. 4). The maximum tip deflection after slewing is less than 0.001 m at $T = 22$ s and 0.0002 m at $T = 34$ s. Taking damping into account does not result in visible distinctions.

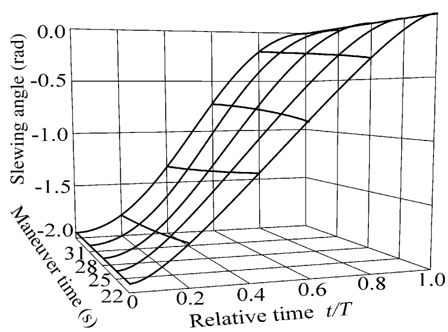

Fig. 2 Optimal slewing angle.

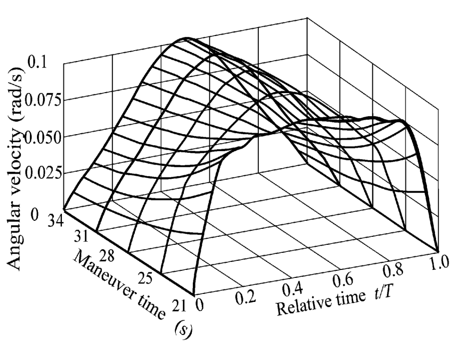

Fig. 3 Optimal slewing angular velocity.

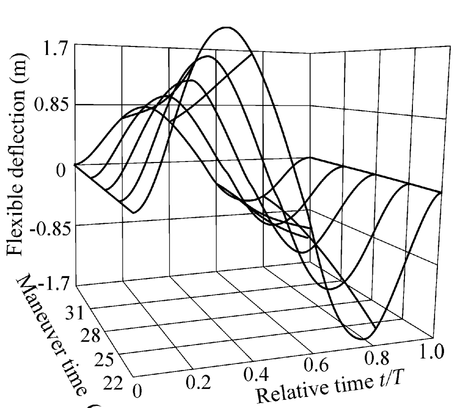

Fig. 4 Flexible deflection of the panel's tip.

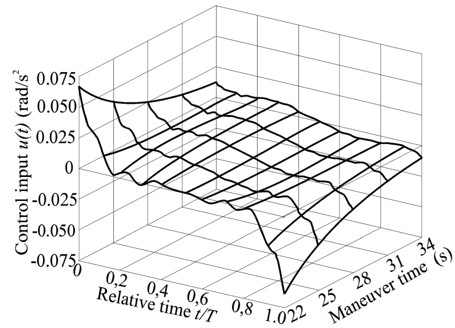

Fig. 5 Angular acceleration of the main body.

The solution of this nonlinear boundary-value problem can require large resources of the onboard computer in some specific cases. Further, the computation of a control program will result in errors because a real slewing control includes the obligatory feedback control. In this connection, computation of the slew program in orbit may be better realized by simpler algorithms using, for example, neural networks technology.

B. Special Case II

The reorientation of one of the spacecraft's principal axes is of specific interest for applications. The problem of determination of the finite rotation axis does not have a unique solution in this case. It is reasonable to choose this axis such that the angle of finite rotation is minimal. This occurs if the axis of finite rotation is orthogonal to the initial and terminal positions of the reoriented axis (i.e., if it lies in one of the principal inertia planes of the unstrained spacecraft). For definiteness, we choose the axis Ox_1 as the reoriented one (see Fig. 1). In this case, the axis of finite rotation yielding the minimal slewing angle lies in the plane x_2Ox_3. Thus, $\gamma_{11} = 0$. The optimal control problems (28–30) become

$$\dot x_1 = x_2; \qquad \dot x_2 = x_3; \qquad \dot x_3 = x_4; \qquad \dot x_4 = v$$
$$\dot x_{2k+3} = x_{2k+4}\dot x_{2k+4} = -n_k x_{2k+4} - \Omega_k^2 x_{2k+3} + x_2^2 x_{2k+3} d_2 + a_k x_3 \qquad (k = 1, \ldots, N) \tag{44}$$

$$I^* = \frac{1}{2}\int_0^T \left[\sum_{k=1}^N \dot x_{2k+4}^2 w_k + v^2 \sum_{k=N+1}^{\infty} \frac{a_k^2 w_k}{\Omega_k^4}\right] \mathrm{d}t \tag{45}$$

$$x_1(T) = \Phi \tag{46}$$

The remaining boundary conditions are homogeneous. The solution of this problem can be only obtained using numerical methods, because of its nonlinearity. The algorithm outlined in the previous section can be applied as well.

The analysis of slewing parameters for real systems shows that the quasi-static deformation conditions (12) are satisfied in some cases for all vibrational modes, starting with the first mode. In the quasi-static case, the analytical solution of the problem under consideration may be obtained. The optimal control problem can be written as follows:

$$\dot x_1 = x_2; \qquad \dot x_2 = x_3; \qquad \dot x_3 = x_4; \qquad \dot x_4 = v \tag{47}$$

$$I^* = \frac{1}{2}\int_0^T v^2 dt \tag{48}$$

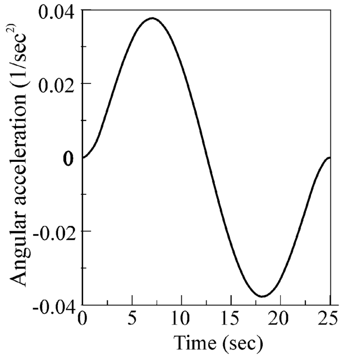

Fig. 6 Angular acceleration of the main body for special case II.

$$x_1(0) = -\Phi; \qquad x_i(0) = 0; \qquad x_j(T) = 0$$
$$(i = 2, 3, 4; j = 1, \ldots, 4) \tag{49}$$

In the absence of constraints, its solution looks like

$$x_i = \sum_{k=i-1}^{7} C_k t^{(k-i+1)} (k - i + 1)!^{-1}$$
$$v = \sum_{k=4}^{7} C_k t^{(k-4)} (k - 4)!^{-1} \tag{50}$$

At boundary conditions (49), the integration constants are, respectively,

$$C_0 = C_1 = C_2 = C_3 = 0; \qquad C_4 = 4!35\Phi T^{-4}$$
$$C_5 = -5!84\Phi T^{-5}; \qquad C_6 = 6!70\Phi T^{-6}; \qquad C_7 = -7!20\Phi T^{-7} \tag{51}$$

The feedforward control (angular acceleration of the main body) for such a case is shown in Fig. 6. It is a smooth curve.

The obtained analytical solutions may be used for the computation of the feedforward control for slewing maneuvers of both sufficiently rigid space vehicles and flexible space vehicles with a high performance of the feedback control. In addition, this solution can be effectively used as a starting approximation for solving the corresponding boundary-value problem of the optimal control for slewing maneuvers of very flexible spacecraft.

IV. Conclusions

A new approach is presented for the formulation of the equations of motion of a spacecraft with attached flexible appendages and for the formulation of the optimal control problem for its reorientation. The approach for derivation of the equations of motion allowed us to establish a finite dimensional system of equations of motion for a continuous mechanical system, which takes into account the interaction of the spacecraft's main body with any number of degrees of freedom of flexible appendages. The approach was developed for the formulation of the problem of optimal reorientation of a spacecraft with flexible appendages. Its use allows one to make the behavior of elastic appendages as close as possible to the quasi-static motion during the slewing maneuver. The slewing maneuver of the space vehicle is thus realized with respect to the Euler axis. The general formulation of the problem and some special cases were considered. For one of them, the numerical solution is obtained using the developed effective algorithm for solving the nonlinear boundary-value problem. For the second special case, it was possible to obtain the analytical solution. In that case, the feedforward control can be found using explicit formulas.

Acknowledgments

The author thanks Hans Troger of the Vienna University of Technology for helpful discussions and comments, Arun Banerjee of the Lockheed Martin Advanced Technology Center for his help in improving the author's writing from an earlier draft of the paper, and particularly the Associate Editor, Panagiotis Tsiotras, whose helpful comments led to a substantial improvement of the paper.

References

[1] Singh, T., "Fuel/Time Optimal Control of the Benchmark Problem," *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 6, 1995, pp. 1225–1231.

[2] Meyer, J. L., and Silverberg, L., "Fuel-Optimal Propulsive Maneuver of an Experimental Structure Exhibiting Spacelike Dynamics," *Journal of Guidance, Control, and Dynamics*, Vol. 19, No. 1, 1996, pp. 141–149.

[3] VanderVelde, W., and He, J., "Design of Space Structure Control Systems Using On–Off Thrusters," *Journal of Guidance, Control, and Dynamics*, Vol. 6, No. 1, 1983, pp. 53–60.

[4] Wie, B., Sinha, R., Sunkel, J., and Cox, K., "Robust Fuel- and Time-Optimal Control of Uncertain Flexible Space Structures," AIAA Guidance, Navigation, and Control Conference, Monterey, CA, AIAA, Washington, D.C., 1993, pp. 939–948; also AIAA Paper 93-3804.

[5] Di Meglio, A., and Ercoli Finzi, A., "Minimum Time Control of Flexible Spacecraft by Hamilton's Principle," *Meccanica*, Vol. 32, No. 6, 1997, pp. 555–565. doi:10.1023/A:1004204510722

[6] Banerjee, A. K., Pedreiro, N., and Singhose, W. E., "Vibration Reduction for Flexible Spacecraft Following Momentum Dumping With/Without Slewing," *Journal of Guidance, Control, and Dynamics*, Vol. 24, No. 3, 2001, pp. 417–427.

[7] Singhose, W., Banerjee, A., and Seering, W., "Slewing Flexible Spacecraft with Deflection-Limiting Input Shaping," *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 2, 1997, pp. 291–298.

[8] Ben-Asher, J., Burns, J. A., and Cliff, E. M., "Time-Optimal Slewing of Flexible Spacecraft," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 2, 1992, pp. 360–367.

[9] Liu, Q., and Wie, B., "Robust Time-Optimal Control of Uncertain Flexible Spacecraft," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 3, 1992, pp. 597–604.

[10] Singh, G., Kabamba, P. T., and McClamroch, N. H., "Planar, Time-Optimal, Rest-to-Rest Slewing Maneuvers of Flexible Spacecraft," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 1, 1989, pp. 71–81.

[11] Singh, T., and Vadali, S. R., "Robust Time-Optimal Control: A Frequency Domain Approach," *Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 2, 1994, pp. 346–353.

[12] Singhose, W., Derezinski, S., and Singer, N., "Extra-Insensitive Input Shapers for Controlling Flexible Spacecraft," *Journal of Guidance, Control, and Dynamics*, Vol. 19, No. 2, 1996, pp. 385–391.

[13] Bainum, P. M., and Li, F., "Optimal Large Angle Maneuvers of a Flexible Spacecraft," *Dynamics of Flexible Structures in Space*, edited by Kirk, C. L., Junkins, J. L., Computational Mechanics Publications, Southampton, and Springer–Verlag, Berlin, 1990.

[14] Pao, L. Y., "Minimum-Time Control Characteristics of Flexible Structures," *Journal of Guidance, Control, and Dynamics*, Vol. 19, No. 1, 1996, pp. 123–129.

[15] Zakrzhevskii, A. E., "Optimal Control for Elastic System as Active Vibrodamping," *Problems of Reliability and Vibroprotection of the Devices*, Irkutsk Polytechnical Inst., Irkutsk, Russia, 1972, pp. 134–139.

[16] Zakrzhevskii, A. E., "Optimal Control for Slewing Maneuver of Free Rigid Body with Flexible Appendages." *Control for Mechanical Systems*, Irkutsk Polytechnical Inst., Irkutsk, Russia, 1977, pp. 4–14.

[17] Farrenkopf, R. L., "Optimal Open Loop Maneuver Profiles for Flexible Spacecraft," *Journal of Guidance and Control*, Vol. 2, No. 6, 1979, pp. 491–498.

[18] Bayo, E., Serna, M. A., Papadopulos, P., and Stube, J., "Inverse Dynamics and Kinematics of Multi-Link Elastic Robots: An Iterative Frequency Domain Approach," *International Journal of Robotics Research*, Vol. 8, No. 6, 1989, pp. 49–92. doi:10.1177/027836498900800604

[19] Singh, T., and Vadali, S. R., "Input-Shaped Control of Three-Dimensional Maneuvers of Flexible Spacecraft," *Journal of Guidance, Control, and Dynamics*, Vol. 16, No. 6, 1993, pp. 1061–1068.

[20] Gorinevsky, D., and Vukovich, G., "Control of Flexible Spacecraft Using Nonlinear Approximation of Input Shape Dependence on Reorientation Maneuver Parameters," *Control Engineering Practice*, Vol. 5, No. 12, 1997, pp. 1661–1671. doi:10.1016/S0967-0661(97)10020-X

[21] Zakrzhevskii, A. E., "Finite-Dimensional Representation of Model of Solid Body with Attached Elastic Elements," *International Applied Mechanics*, Vol. 19, No. 1, 1983, pp. 85–91.

[22] Lurie, A. I., *Analytical Mechanics*, Springer, New York, 2002.

[23] Balchen, T. G., *Dynamic Optimization of Continuous Processes*, Institutt for Reguleringsteknikk, Norges Tekniske Høgskole, Trondheim, Norway, 1961, p. 150.

[24] Mitter, S. K., "Successive Approximation Methods for the Solution of Optimal Control Problems," *Automatica*, Vol. 3, Nos. 3–4, 1966, pp. 135–149. doi:10.1016/0005-1098(66)90009-4

[25] Martin, G. D., Bryson, A. E., Jr., "Attitude Control of Flexible Spacecraft," *Journal of Guidance and Control*, Vol. 3, No. 1, 1980, pp. 37–41.